Derivatives of Trigonometric Functions

(2.12) Theorem. (i) If $y = \sin x$ (x measured in radians), then

$$\frac{dy}{dx} = \cos x.$$

Proof. We have

$$\begin{aligned} y + \Delta y &= \sin(x + \Delta x) \\ \Delta y &= \sin(x + \Delta x) - \sin x \\ &= \sin x \cos \Delta x + \cos x \sin \Delta x - \sin x \\ &= \sin x(\cos \Delta x - 1) + \cos x \sin \Delta x \end{aligned}$$

Therefore, $$\frac{\Delta y}{\Delta x} = \sin x\left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \cos x \frac{\sin \Delta x}{\Delta x}$$

Taking limits as $\Delta x \to 0$, we get

$$\frac{dy}{dx} = (\sin x) \cdot 0 + (\cos x) \cdot 1,$$

since $\displaystyle\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} = 0$, by Example 30 of Chapter 1

and $\displaystyle\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$, by Example 31 of Chapter 1

Thus $\dfrac{d}{dx} \sin x = \cos x$.

(ii) The derivative of $y = \cos x$ is $\dfrac{dy}{dx} = -\sin x$

Proof. Here

$$\begin{aligned} y + \Delta y &= \cos(x + \Delta x) \\ \Delta y &= \cos(x + \Delta x) - \cos x \\ &= \cos x \cos \Delta x - \sin x \sin \Delta x - \cos x \\ &= \cos x(\cos \Delta x - 1) - \sin x \sin \Delta x \end{aligned}$$

Therefore, $$\frac{\Delta y}{\Delta x} = \cos x\left(\frac{\cos \Delta x - 1}{\Delta x}\right) - \sin x \frac{\sin \Delta x}{\Delta x}$$

Proceeding to limits as $\Delta x \to 0$, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \cos x \,.\, 0 - \sin x \,.\, 1 \\ &= -\sin x. \end{aligned}$$

For other trigonometric functions, one can easily derive the following:

(iii) $\frac{d}{dx} \tan x = \sec^2 x$

(iv) $\frac{d}{dx} \cot x = -\csc^2 x$

(v) $\frac{d}{dx} \sec x = \sec x \tan x$

(vi) $\frac{d}{dx} \csc x = -\csc x \cot x.$

Inverse Trigonometric Functions

(2.13) Theorem. In the open interval $]-1, 1[$, the function $y = \arcsin x$ is differentiable and

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$$

Proof. If $y = \arcsin x$, then $x = \sin y$ so that

$$\frac{dx}{dy} = \cos y$$

or $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-\sin^2 y}}, \quad (\cos y \neq 0)$$

The sign of the radical is to be the same as that of $\cos y$. By definition,

$$\frac{-\pi}{2} < \arcsin x < \frac{\pi}{2}$$

i.e., $\frac{-\pi}{2} < y < \frac{\pi}{2}$, so that $\cos y$ is positive.

Hence $$\frac{dy}{dx} = \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$

The reader should derive the following formulas by the same method that has been used in finding the derivative of $\arcsin x$.

(i) $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}, \ |x| < 1$

(ii) $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

(iii) $\frac{d}{dx} \operatorname{arccot} x = \frac{-1}{1+x^2}$

(iv) $\dfrac{d}{dx}\ \text{arcsec}\, x = \dfrac{1}{x\sqrt{x^2-1}}, \ |x| > 1$

(v) $\dfrac{d}{dx}\ \text{arccsc}\, x = \dfrac{-1}{x\sqrt{x^2-1}}, \ |x| > 1.$

Logarithmic and Exponential Functions

(2.14) Theorem. If $y = \log_a x$, $(x > 0, a > 1)$ then

$$\frac{dy}{dx} = \frac{1}{x}\log_a e$$

Proof. By definition,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\log_a (x+h) - \log_a x}{h}$$

$$= \lim_{h \Rightarrow 0} \frac{\log_a \left(\dfrac{x+h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \log_a \left(1 + \frac{h}{x}\right)$$

$$= \lim_{h \to 0} \frac{1}{x} \cdot \frac{x}{h} \log_a \left(1 + \frac{h}{x}\right)$$

$$= \lim_{h \to 0} \frac{1}{x} \lim_{h \to 0} \log_a \left(1 + \frac{h}{x}\right)^{x/h}$$

$$= \frac{1}{x} \log_a \left[\lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{x/h}\right]$$

$$= \frac{1}{x} \log_a e, \text{ by Corollary of Example 28, Chapter 1}$$

$$= \frac{1}{x \log_e a} = \frac{1}{x \ln a},$$ where $\ln a$ denotes the natural logarithm of a.

If $a = e$, then $y = \log_e x = \ln x$

so that $$\frac{dy}{dx} = \frac{d}{dx}\ln x$$

$$= \frac{1}{x \ln e} = \frac{1}{x}.$$

Corollary. If $f(x) = \ln|x|$, then $f'(x) = \frac{1}{x}$, for all $x \in R - \{0\}$.

Proof. If $x > 0$, then $|x| = x$ and in this case

$$f'(x) = \frac{d}{dx}\ln x = \frac{1}{x}.$$

If $x < 0$, then $|x| = -x$ and therefore

$$f'(x) = \frac{d}{dx}\ln(-x) = \frac{(-1)}{(-x)}, \text{ by the Chain Rule}$$

$$= \frac{1}{x}.$$

Thus $\frac{d}{dx}\ln|x| = \frac{1}{x}$, $x \neq 0$.

(2.15) Theorem. If $y = a^x$, then $\frac{dy}{dx} = a^x \ln a$.

Proof.
$$\frac{dy}{dx} = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0}\left(a^x \cdot \frac{a^h - 1}{h}\right)$$

$$= \lim_{h\to 0} a^x \cdot \lim_{h\to 0}\frac{a^h - 1}{h}$$

$= a^x \log_e a$, by **Example 29 of Chapter 1.**

$$= a^x \ln a.$$

Corollary. If $y = e^x$, then

$$\frac{dy}{dx} = e^x \ln e = e^x.$$

Thus e^x remains invariant under the operation of differentiation.